

# Closed-form solutions of the Wheeler-DeWitt equation in a scalar-vector field cosmological model by Lie symmetries

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We apply as selection rule to determine the unknown functions of a cosmological model the existence of Lie point symmetries for the Wheeler-DeWitt equation of quantum gravity. Our cosmological setting consists of a flat Friedmann-Robertson-Walker metric having the scale factor  $a(t)$ , a scalar field with potential function  $V(\phi)$  minimally coupled to gravity and a vector field of its kinetic energy is coupled with the scalar field by a coupling function  $f(\phi)$ . Then, the Lie symmetries of this dynamical system are investigated by utilizing the behavior of the corresponding minisuperspace under the infinitesimal generator of the desired symmetries. It is shown that by applying the Lie symmetry condition the form of the coupling function and also the scalar field potential function may be explicitly determined so that we are able to solve the Wheeler-DeWitt equation. Finally, we show how we can use the Lie symmetries in order to construct conservation laws and exact solutions for the field equations.

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## 1. INTRODUCTION

Symmetries have always played a central role in conceptual discussion of the classical and quantum physics. The main reason may be that various laws of conservation, such as energy, momentum, angular momentum, etc., that provide the integrals of motion for a given dynamical system, are indeed the result of existence of some kinds of symmetry in that system. From a more general point of view, it can be shown that all such conservation laws are particular cases of the so-called Noether theorem, according to which for every one-parameter group of transformation on the configuration space of a system, which act on the Lagrangian  $\mathcal{L}$  and preserves the Action integral, i.e., the Euler-Lagrange equations  $E_L(\mathcal{L})$ , there exists a first integral of motion [1–4]. In mathematical language this means that if the vector field  $X = \xi(\tau, q^C) \partial_\tau + \eta^A(\tau, q^C) \partial_C$  is the generator of the above diffeomorphism, then there exist a function  $g$  such as [2, 3]

$$L_X \mathcal{L} + \mathcal{L} \frac{d\xi}{d\tau} = \frac{dg}{d\tau}. \quad (1)$$

Numerous applications of Noether's theorem<sup>1</sup> in general relativity and cosmology are those concerned with the following form of action [6–10]

$$\mathcal{S} = \int_{\mathcal{M}} d\tau \left[ \frac{1}{2} \mathcal{G}_{AB} \frac{dq^A}{d\tau} \frac{dq^B}{d\tau} - \mathcal{U}(\mathbf{q}) \right], \quad (2)$$

where  $q^A$  are the coordinates of the configuration space with metric  $\mathcal{G}_{AB}$  (the indices  $A, B, \dots$  run over the dimension of this space),  $\mathcal{U}(\mathbf{q})$  is the potential function and  $\tau$  is an affine parameter along the evolution path of the system. In time-parameterized theories such as general relativity, the action retains its form under time reparameterization. Therefore, one may relate the affine parameter  $\tau$  to a time parameter  $t$  by a lapse function  $N(t)$  through  $N dt = d\tau$ .

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<sup>1</sup> Recently in the literature a vector field which satisfies the condition (1) has been termed as “Noether Gauge Symmetry”, see [5]. This is incorrect terminology, since condition (1) is that which has been introduced by E. Noether in her original work. The function,  $g$ , of (1) is a boundary term (not a gauge function) introduced to allow for the infinitesimal transformations which in the value of the Action Integral produced by the infinitesimal change in the boundary of the domain caused by the infinitesimal transformation of the variables in the Action Integral.

In these cases the action (2) can be written as

$$\mathcal{S} = \int_{\mathcal{M}} dt \mathcal{L}(q^A, \dot{q}^A) = \int_{\mathcal{M}} dt N \left[ \frac{1}{2N^2} \mathcal{G}_{AB} \dot{q}^A \dot{q}^B - \mathcal{U}(\mathbf{q}) \right], \quad (3)$$

where an over-dot indicates derivation with respect to the time parameter  $t$  and  $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}})$  is the Lagrangian function of the system. A straightforward calculation based on the Hamiltonian formalism leads us to the Hamiltonian constraint

$$H = N \left[ \frac{1}{2} \mathcal{G}^{AB} P_A P_B + \mathcal{U}(\mathbf{q}) \right] = N\mathcal{H} \equiv 0, \quad (4)$$

where  $P_A$  is the momentum conjugate to  $q^A$ . Therefore, under canonical quantization, the above constraint yields the Wheeler-DeWitt (WDW) equation  $\mathcal{H}\Psi(\mathbf{q}) = 0$ , where  $\Psi(\mathbf{q})$  is the wave function of the quantized system and  $\mathcal{H}$  should be written in a suitable operator form. If one makes a natural choice of factor ordering, the WDW equation may be written as

$$\mathcal{H}\Psi(\mathbf{q}) = [\Delta + \mathcal{U}(\mathbf{q})] \Psi(\mathbf{q}) = 0, \quad (5)$$

where  $\Delta = \frac{1}{\sqrt{-\mathcal{G}}} \partial_A (\sqrt{-\mathcal{G}} \mathcal{G}^{AB} \partial_B)$  is the Laplacian operator in the space with metric  $\mathcal{G}_{AB}$ . There is also a natural choice of factor ordering which gives the conformally invariant WDW equation as [11]

$$\mathcal{H}\Psi = \left[ \frac{1}{2} \Delta + \frac{n-2}{8(n-1)} \mathcal{R} + \mathcal{U}(\mathbf{q}) \right] \Psi(\mathbf{q}) = 0, \quad (6)$$

where  $\mathcal{R}$  is the Ricci scalar obtained from the metric  $\mathcal{G}_{AB}$ . In general, in terms of the 3 + 1 decomposition notation, the WDW equation which comes from the Hamiltonian constraint has the form

$$\mathcal{H}\Psi = \left[ -4\kappa^2 \mathcal{G}_{ijkl} \frac{\delta^2}{\delta h_{ij} \delta h_{kl}} + \frac{\sqrt{h}}{4\kappa^2} (-\mathcal{R} + 2\Lambda + 4\kappa^2 T^{00}) \right] \Psi = 0, \quad (7)$$

in which

$$\mathcal{G}_{ijkl} = \frac{1}{2\sqrt{h}} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl}), \quad (8)$$

is the the metric of superspace (the space of all 3-geometries, with metric  $h_{ij}$  and Ricci scalar  $\mathcal{R}$ , and the matter configuration  $\Phi$ ). For a scalar field, for instance, we have

$$T^{00} = -\frac{1}{2h} \frac{\delta^2}{\delta \Phi^2} + \frac{1}{2} h^{ij} \Phi_{,i} \Phi_{,j} + V(\Phi). \quad (9)$$

Note that the WDW equation (7) is not a single differential equation. In fact, by which one has one equation in each point of the 3-dimensional hypersurfaces. This means that (7) is in general a hyperbolic functional differential equation on superspace. However, in the case of the minisuperspace approximation in which we will truncate (by applying the symmetries) the infinite degrees of freedom of the superspace to a finite number, instead of having a WDW equation for each point of the spatial hypersurface, we have a single WDW equation for all of them.

In this work we apply a geometric selection rule for the determination of the potential  $\mathcal{U}(\mathbf{q})$  of the action integral (3) which depends on the Lie symmetries of the WDW equation (6). This selection rule was proposed in [12] in order to find the exact solution of a hyperbolic scalar field cosmological model in the presence of a perfect fluid with constant equation of state parameter. In [12] it has been shown that the existence of a Lie symmetry for the WDW equation is equivalent with the existence of a conservation law for the field equations. That means that Lie symmetries can be used in order to find invariant solutions for the WDW equation, and study the integrability of the field equations. The plan of the paper is as follows.

In section 2 we consider a spatially flat Friedmann–Robertson–Walker (FRW) space-time with scalar and vector fields in a minisuperspace point of view. The minisuperspace variables turn out to correspond to the scale factor of the universe, a scalar field minimally coupled to gravity and a vector field coupled to scalar field by its kinetic energy term. Such a model is studied in [14] to investigate the possible isotropic power-law inflationary scenarios in the framework of gauge field models. Section 3 is devoted to apply the Lie symmetry method by means of which we determine the transformations which leave the WDW equation invariant. The application of the Lie invariants are given in section 3.2, where we determine all the possible solutions to the WDW equation. In section 4, we use the Lie symmetries of the WDW equation to construct Noetherian conservation laws for the field equations. Furthermore we study the integrability of the field equations and we determine the exact cosmological solution of the model. Finally in section 5 we summarize the results and draw our conclusions.

## 2. THE MODEL

An important issue in all of the model theories related to cosmology is to select the matter or any extra fields used to couple with the gravitational part of the model's action. From the standpoint of the theory of inflation, the most widely used field has traditionally been the scalar field. Such a field are playing an increasingly important role in the recent cosmological models. Maybe, the main reason is that a scalar field does not carry any internal or external index and so, it makes it somewhat easy to work. Another field which has seldom been studied in the literature is the vector field (or even more general gauge fields). In general, theories studying gauge fields coupled to gravity result in Einstein-Maxwell (or Einstein-Yang-Mills) system which in comparison with the case of a scalar field are not easy to deal. However, the cosmological aspects of such systems have been studied in a few cases by a number of works, see for instance [13] and the references therein. Here, we pick a step further and consider a model in which in addition of a scalar field, a vector field is also present by a special kind of coupling with scalar field. The importance of gauge fields when it becomes clear that we are going to explain anisotropic inflation at high energy levels. However, as is shown in [14], for special exponential forms for the scalar field's potential and its coupling function with the vector field, such models can also produce isotropic power-law inflation. By this motivations we consider a gravity model whose dynamics is given by the following action [14, 15]

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2}R - \frac{1}{2}g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) - \frac{1}{4}f^2(\phi) F_{\mu\nu} F^{\mu\nu} \right], \quad (10)$$

where  $\phi(t)$  is a homogeneous scalar field minimally coupled to gravity,  $V(\phi)$  is the potential function, and  $F_{\mu\nu}$  is the strength tensor of the vector field  $A_\mu$  with standard definition  $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ . As the action shows, a coupling function  $f(\phi)$  couples the vector and scalar fields to each other. In such a gravity model let us consider a homogeneous and isotropic cosmological model in which the space-time is assumed to be of flat FRW whose line element can be written

$$ds^2 = -N^2(t) dt^2 + a^2(t) (\delta_{ij} dx^i dx^j), \quad (11)$$

where  $N(t)$  and  $a(t)$  are the lapse function and the scale factor, respectively. To go forward, we introduce a homogeneous vector field in the direction of the  $z$ -axis:  $A_\mu = (0; 0, 0, \zeta(t))$ , by means of which one immediately gets  $F_{\mu\nu} F^{\mu\nu} = -2N^{-2}a^{-2}\dot{\zeta}^2$ , see also [14]. We have assumed that the scalar field  $\phi$ , and the vector field  $A_\mu$ , inherits the symmetries of the underlying spacetime.

With the above results at hand, the action (10) can be written in the point-like form  $S = \int dt L(\mathbf{q}, \dot{\mathbf{q}})$ , where  $\mathbf{q} = (a, \phi, \zeta)$  are the coordinates of the configuration space and

$$\mathcal{L}(a, \dot{a}, \phi, \dot{\phi}, \zeta, \dot{\zeta}) = \frac{1}{N} \left( -3a\dot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2 + \frac{1}{2}af^2(\phi)\dot{\zeta}^2 \right) - a^3NV(\phi), \quad (12)$$

is an effective Lagrangian variation of which with respect to  $\mathbf{q}^A = (a, \phi, \zeta)$ , yields the corresponding Euler-Lagrange equations for the dynamics of the model. Furthermore, in comparison with (3), from the above Lagrangian we have that the minisuperspace is,

$$\mathcal{G}_{AB} = \text{diag}(-6a, a^3, af^2(\phi)), \quad (13)$$

the potential is,  $\mathcal{U}(\mathbf{q}) = a^3V(\phi)$  and the Ricci scalar  $\mathcal{R}$  is given from the following expression

$$\mathcal{R} = \frac{a^{-3}f^{-1}}{12} (f - 24f_{,\phi\phi}). \quad (14)$$

By applying the standard Legendre transformation on the Lagrangian (12) we are arrived at the following form for the Hamiltonian function

$$H = N\mathcal{H} = N \left( -\frac{p_a^2}{12a} + \frac{p_\phi^2}{2a^3} + \frac{p_\zeta^2}{2af^2} + a^3V(\phi) \right), \quad (15)$$

in which we have used the momenta conjugate to the dynamical variables as

$$p_a = -\frac{6}{N}a\dot{a}, \quad p_\phi = \frac{1}{N}a^3\dot{\phi}, \quad p_\zeta = \frac{1}{N}af^2\dot{\zeta}. \quad (16)$$

Now, we can construct the Hamiltonian equations of motion  $\dot{\mathbf{q}} = \{\mathbf{q}, H\}$  which are nothing but the set of equations (16) and  $\dot{\mathbf{p}}_q = \{\mathbf{p}_q, H\}$  which are

$$\dot{p}_a = -\frac{1}{12} \frac{p_a^2}{a^2} + \frac{3}{2} \frac{p_\phi^2}{a^4} + \frac{1}{2} \frac{p_\zeta^2}{a^2 f^2} - 3a^2 V, \quad (17)$$

$$\dot{p}_\phi = \frac{f, \phi}{a f^3} p_\zeta^2 - a^3 V_{, \phi}, \quad \dot{p}_\zeta = 0. \quad (18)$$

The quantum version of this model may be described by the WDW equation written in the form of relation (6) which for our model takes the form [17]

$$\left[ \Delta + \frac{n-2}{4(n-1)} R_{(\gamma)} + 2Na^3 V(\phi) \right] \Psi = 0, \quad (19)$$

where  $\Delta = \frac{1}{\sqrt{-\gamma}} \frac{\partial}{\partial x^A} (\sqrt{-\gamma} \gamma^{AB} \frac{\partial}{\partial x^B})$ ,  $R_{(\gamma)}$  and  $n$  are: the Laplace operator, the Ricci scalar and the dimension of the minisuperspace  $\gamma_{AB}$  respectively, in which  $\gamma_{AB}$ , is the conformally related with the minisuperspace  $\mathcal{G}_{AB}$ , that is,  $\gamma_{AB} = N^{-1} \mathcal{G}_{AB}$ . Hence the line element of  $\gamma_{AB}$  is

$$ds_\gamma^2 = N^{-1} (-6ada^2 + a^3 d\phi^2 + af^2(\phi) d\zeta^2), \quad (20)$$

where  $N = N(a, \phi, \zeta)$ .

The application of Noether's theorem of this model has been studied recently in [15], which some late time accelerated classical solutions are obtained. In the following, we are going to consider the quantum aspects of the problem by applying the Lie symmetry method on equation (19).

### 3. POINT SYMMETRIES AND INVARIANT FUNCTIONS

For the convenience of the reader in the following lines we give the basic properties and definitions of the point symmetries of differential equations and the application of the group invariants.

Consider the partial differential equation (PDE)  $\Theta = \Theta(q^A, \Psi, \Psi_{,A}, \Psi_{,AB})$ ,  $\Theta = 0$ , which is invariant under the action of the one parameter point transformation

$$\bar{q}^A = q^A + \varepsilon \xi^A(q^B, \Psi), \quad \bar{\Psi} = \Psi + \varepsilon \eta(q^B, \Psi), \quad (21)$$

that means, that there exists a function  $\Lambda$  such that the following condition holds [2]

$$\mathbf{X}^{[2]}(\Theta) = \Lambda \Theta, \quad (22)$$

or equivalently

$$X^{[2]}(\Theta) = 0 \quad (23)$$

where

$$\mathbf{X}^{[n]} = \mathbf{X} + \eta_{[A]} \partial_{\Psi_{,A}} + \eta_{[AB]} \partial_{\Psi_{,AB}}, \quad (24)$$

is the  $n$ -th jet prolongation vector of the generator  $\mathbf{X}$  of the one parameter point transformation (21) [4]. This generator which may be written in the form

$$\mathbf{X} = \frac{\partial \bar{q}^A}{\partial \varepsilon} \partial_{q^A} + \frac{\partial \bar{\Psi}}{\partial \varepsilon} \partial_\Psi \quad (25)$$

and it is called, Lie point symmetry of the PDE  $H$  [2, 4]. The importance of Lie point symmetries of differential equations is that it can be used in order to determine invariant solutions or transform solutions to solutions (see [2] for details).

In what follows, we use the Lie symmetries of the WDW equation (19) as a selection rule in order to determine the unknown functions of the action (10). Furthermore, we apply the zero order invariants in order to find invariant solution of the WDW equation (19) and exact solution of the field equations (16)-(18).

### 3.1. Lie symmetries of the WDW equation

In order to determine the Lie and the Noether symmetries of the WDW equation we will follow the results of [16] which relate the point symmetries of the WDW equation with the conformal algebra of the space which defines the Laplace operator. This means that we can separate the problem in two steps: (a) we will study the conformal algebra of the minisuperspace and (b) we will determine the unknown potential.

Specifically, it has been shown that a vector field  $X$  is a Lie point symmetry for the WDW equation (19), if and only if,  $X$  is a Conformal Killing vector (CKV) of the space which defines the Laplace operator  $\Delta$ , that is  $X_{(A;B)} = \psi(q^A) \gamma_{AB}$ , and the following condition holds

$$X^A (\mathcal{U}(\mathbf{q}))_{;A} + 2\psi\mathcal{U}(\mathbf{q}) = 0. \quad (26)$$

For a general function  $f(\phi)$  the minisuperspace (20) admits a two dimensional conformal algebra characterized by the vector fields  $X_1 = \partial_\zeta$  and  $X_H = \frac{1}{3}a\partial_a + \frac{1}{3}\zeta\partial_\zeta$ . Hence from condition (26) it is easy to see that while the vector field  $X_1$  generates a point symmetry for the WDW equation (19) regardless of the form of the potential  $V(\phi)$ , such a non zero potential for which  $X_H$  generates a symmetry vector does not exist.

Now, let us to see if it is possible to choose the function  $f(\phi)$  in such a way that the minisuperspace (20) admits a greater conformal algebra. In particular, we know that the minisuperspace (20) may has a ten dimensional conformal algebra when it is conformally flat. On the other hand, it is well known that the three dimensional space-times are conformally flat if and only if the Cotton-York tensor vanishes. Therefore, by applying this statement on (20) we are led to the following system

$$f_{,\phi\phi} - \frac{1}{f}f_{,\phi}^2 = 0, \quad f_{,\phi\phi\phi} - \frac{f_{,\phi}}{f}f_{,\phi\phi} = 0, \quad (27)$$

which leads to the general solution <sup>2</sup>

$$f(\phi) = f_0 \exp(\omega\phi). \quad (28)$$

Furthermore, space (20) admits the same conformal algebra with  $\mathcal{G}_{AB}$ , (12) which is conformal related with the  $(1+2)$  decomposable spacetime,

$$d\bar{s}^2 = \frac{1}{af^2(\phi)} (-6ada^2 + a^3d\phi^2) + d\zeta^2. \quad (29)$$

The last space, for arbitrary  $f(\phi)$ , admits the two CKVs  $X_1, X_H$ . However according to [18], admits extra CKVs if and only if the two dimensional space

$$d\bar{s}_{(2)}^2 = \frac{1}{af^2(\phi)} (-6ada^2 + a^3d\phi^2) \quad (30)$$

is flat, or admits gradient CKVs, the last means that (30) is a space of constant curvature. Hence, from the last line element we calculate the Ricci scalar,  $R_{(2)} = 2a^{-2} (f_{,\phi\phi}f - f_{,\phi}^2)$ , from where we can see that  $R_{(2)} = 0$ , when  $f(\phi)$  is given by (28), and there does not exist any function  $f(\phi)$ , where  $R_{(2)} = \text{const} \neq 0$ . Recall that all two dimensional spaces are Einstein spaces and conformally flat. Hence we conclude that the WDW equation (19) can admits Lie symmetries, if and only if,  $f(\phi)$  is given by (28).

Thus, the minisuperspace takes the form

$$ds_{\bar{\gamma}}^2 = a^{-2\sqrt{6}\omega} (-6da^2 + a^2d\phi^2 + e^{2\omega\phi}d\zeta^2), \quad (31)$$

in which without loss of generality we have chosen  $N = a^{1+2\sqrt{6}\omega}$ , so that the minisuperspace becomes Ricci-flat, i.e.  $R_{(\gamma)} = 0$ . Recall that the WDW equation (6) is defined by the conformal invariant Laplace operator  $\hat{L}_\gamma = \Delta_\gamma + \frac{n-2}{4(n-1)}R_\gamma$ , and that under a conformal transformation,  $\bar{\gamma}_{AB} = e^{2\Omega}\gamma_{AB}$ , holds,  $\hat{L}_{\bar{\gamma}}(\Psi) = e^{-\frac{n+2}{2}\Omega}\hat{L}_\gamma(e^{\frac{n-2}{2}\Omega}\Psi)$ . Moreover the symmetry analysis is independent on the conformal factor  $N$  [16].

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<sup>2</sup> In the following we consider  $\omega \neq 0$  and  $f_0 = 1$ .

TABLE I: The commutators of the Lie algebra  $G_2$ 

$[X_A, X_B]$	$X_1$	$X_2$	$X_3$	$X_\Psi$
$X_1$	0	$(\frac{1}{6\omega} - \omega) X_1$	$2\omega X_2 - \frac{\sqrt{6\omega+1}}{6} X_\Psi$	0
$X_2$	$-(\frac{1}{6\omega} - \omega) X_1$	0	$(\frac{1}{6\omega} - \omega) X_3$	0
$X_3$	$-2\omega X_2 + \frac{\sqrt{6\omega+1}}{6} X_\Psi$	$-(\frac{1}{6\omega} - \omega) X_3$	0	0
$X_\Psi$	0	0	0	0

TABLE II: The commutators of the Lie algebra  $G_3$ 

$[X_A, X_B]$	$X_1$	$X_2$	$\bar{X}_3$	$X_4$	$X_\Psi$
$X_1$	0	$-\frac{\sqrt{6}}{9} X_1$	$X_4$	0	0
$X_2$	$-\frac{\sqrt{6}}{9} X_1$	0	$\frac{5\sqrt{6}}{18} X_3$	$\frac{7\sqrt{6}}{18} X_4$	0
$\bar{X}_3$	$-X_4$	$-\frac{5\sqrt{6}}{18} X_3$	0	0	0
$X_4$	0	$-\frac{7\sqrt{6}}{18} X_4$	0	0	0
$X_\Psi$	0	0	0	0	0

Therefore with the use of (31) the WDW equation (19) can be written as

$$\left( -\frac{1}{6} \Psi_{,aa} + a^{-2} \Psi_{,\phi\phi} + e^{-2\omega\phi} \Psi_{,\zeta\zeta} \right) + a^{-1} \left( -\frac{(1+\sqrt{6}\omega)}{6} \Psi_{,a} + \frac{\omega}{a} \Psi_{,\phi} \right) + 2a^4 V(\phi) \Psi = 0. \quad (32)$$

Following [16], we conclude that when (we consider  $V_\phi \neq 0$ )

$$V(\phi) = V_0 e^{-\lambda\phi} \quad (33)$$

equation (32) is invariant under the Lie algebra  $G_1 = \text{span} \{X_1, X_2, X_\Psi\}$  where<sup>3</sup>

$$X_1 = \partial_\zeta, \quad X_2 = \frac{\lambda}{6} a \partial_a + \partial_\phi + \left( \frac{\lambda}{6} - \omega \right) \zeta \partial_\zeta, \quad X_\Psi = \partial_\Psi. \quad (34)$$

The only non-zero commutator of the above vector fields is  $[X_1, X_2] = (\frac{\lambda}{6} - \omega) X_1$ .

However, for special values of the constants  $\omega, \lambda$  the WDW equation (32) may admit some extra symmetries. In particular we have the following two cases, (I):  $\lambda = \omega^{-1}$ , and, (II):  $\lambda = 4\omega$ , with  $\omega = \frac{\sqrt{6}}{3}$ .

For the case (I), equation (32) is invariant under the action of the Lie algebra  $G_2 = \text{span} \{X_1, X_2, X_3, X_\Psi\}$  where

$$X_3 = \frac{a\zeta}{3} \partial_a + 2\omega\zeta \partial_\phi + \left( \frac{1-6\omega^2}{6} \zeta^2 + a^2 e^{-2\omega\phi} \right) \partial_\zeta - \frac{\sqrt{6}\omega+1}{6} \zeta X_\Psi, \quad (35)$$

whereas for the case (II), the WDW equation is invariant under the action of the Lie algebra  $G_3 = \text{span} \{X_1, X_2, \bar{X}_3, X_4, X_\Psi\}$ , in which

$$\bar{X}_3 = a\zeta e^{\frac{\sqrt{6}}{6}\phi} \left( \frac{\sqrt{6}}{6} a \partial_a + \partial_\phi \right) + \frac{\sqrt{6}}{3} a^3 \sqrt{6} e^{-\frac{\sqrt{6}}{2}\phi} \partial_\zeta \quad (36)$$

and

$$X_4 = a e^{\frac{\sqrt{6}}{6}\phi} (a \partial_a + \partial_\phi). \quad (37)$$

The commutators of the Lie algebras  $G_2$ , and  $G_3$ , are given in tables I, and II.

In the following, we consider the application of the Lie algebras  $G_1$ ,  $G_2$ , and  $G_3$ , in order to determine the invariant solutions of the WDW equation.

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<sup>3</sup> There is also the Lie symmetry  $X_B = B(a, \phi, \zeta) \partial_\Psi$ , where  $B(a, \phi, \zeta)$  is a solution of (32). However since  $X_B$  is a trivial symmetry we will omit it.

### 3.2. Invariant solutions of the WDW equation

The WDW equation (32) is a second order PDE in which the wave function depends on the variables  $\{a, \phi, \zeta\}$ . This means that in order to reduce this equation to the form of an ordinary differential equation (ODE) we need the application of two symmetry vectors [19]. To do this, we consider the invariance of the WDW equation under the act of the: (A) Lie algebra  $G_1$ , (B) Lie algebra  $G_2$ , and (C) Lie algebra  $G_3$ . For each case we obtain the analytical solution of equation (32).

We would like to remark that it is possible to determine an group invariant solution one symmetry vector, for instance see [20]. The reason for that is that the application of a Lie symmetry in a PDE leads to a new differential equation, which in general is independent from the original equation, that is, new Lie symmetries can arise, a class of these symmetries are termed as hidden symmetries, for a discussion see [21, 22] and references therein.

#### 3.2.1. Case A: Lie invariants of $G_1$

For an exponential potential function such as (33) and arbitrary values of the constants  $\lambda, \omega$ , the commutation relations of the Lie algebra  $G_1$  allow to apply the Lie invariants of the subalgebras  $A_{(i)} = \{X_1, X_2\}$  and  $A_{(ii)} = \{X_1, X_2 + \beta\Psi\partial_\Psi\}$ ,  $\beta \in \mathbb{C}$ . By applying the zero order invariants of  $A_{(i)}$  on (32) we get the wave function as<sup>4</sup>

$$\Psi_{A_{(i)}}(a, \phi, \zeta) = \Phi(w) \quad , \quad w = \phi - \frac{6}{\lambda} \ln a, \quad (38)$$

where  $\Phi(w)$  satisfies the following ODE

$$(\lambda^2 - 6) \Phi_{,ww} + \omega\lambda \left( \sqrt{6} + \lambda \right) \Phi_{,w} + 2\lambda^2 V_0 e^{-\lambda w} \Phi = 0. \quad (39)$$

Repeating this procedure but this time with the help of the Lie invariants of  $A_{(ii)}$  we find the wave function

$$\Psi_{A_{(ii)}}(a, \phi, \zeta) = \Phi(w) a^{\frac{6\beta}{\lambda}} \quad , \quad w = \phi - \frac{6}{\lambda} \ln a, \quad (40)$$

where now the function  $\Phi(w)$  should be a solution of the following ODE

$$\begin{aligned} 0 = & (\lambda^2 - 6) \Phi_{,ww} + \left( 12\beta + \omega\lambda \left( \sqrt{6} + \lambda \right) \right) \Phi_{,w} \\ & - \left( 6\beta^2 + \sqrt{6}\omega\beta\lambda - 2\lambda^2 V_0 e^{-\lambda w} \right) \Phi. \end{aligned} \quad (41)$$

It is seen that the solution (38) is a special case of  $A_{(ii)}$  for  $\beta = 0$ . Now, if we take the numerical value  $\lambda = \pm\sqrt{6}$ , the solutions of (41) can be expressed as follows

$$\Phi(w) = \Phi_0 \exp \left( \pm \frac{(\pm 3\beta^2 + 3\omega\beta)w + \sqrt{6}V_0 e^{\mp\sqrt{6}w}}{6(\omega + \beta)} \right), \quad (42)$$

for which we are led to the following wave function as a solution of (32)

$$\Psi_{A_{(ii)}}(w, \zeta) = \Phi_0 a^{\frac{6\beta}{\lambda}} \exp \left( \pm \frac{(\pm 3\beta^2 + 3\omega\beta)w + \sqrt{6}V_0 e^{\mp\sqrt{6}w}}{6(\omega + \beta)} \right). \quad (43)$$

*a. Subcase A1:*  $\lambda = 6\omega$  Let us now have a glance at the special case in which  $\lambda = 6\omega$ . This case is especially important in the sense that all of the generators of the Lie algebra  $G_1$  commute with each other since we get  $[X_1, X_2] = 0$ . Therefore, we may apply on equation (32) some additional invariants of the Lie algebras  $A_{(iii)} = \{X_1 + \gamma\Psi\partial_\Psi, X_2\}$  and  $A_{(iv)} = \{X_1 + \gamma\Psi\partial_\Psi, X_2 + \beta\Psi\partial_\Psi\}$ . We continue with the application of  $A_{(iv)}$  since when  $\beta = 0$ ,  $A_{(iii)} = A_{(iv)}$ . With the same steps as in the previous section we will have

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<sup>4</sup> The symbolic package Sym for Mathematica have been used to test the results [23].

$$\Psi_{A_{(iv)}}(a, \phi, \zeta) = \Phi(w) a^{\frac{\beta}{\omega}} e^{\gamma \zeta}, \quad w = \phi \mp \sqrt{6} \ln a, \quad (44)$$

in which

$$\begin{aligned} 0 = & (6\omega^2 - 1) \Phi_{,ww} + \left( 2\beta + \sqrt{6}\omega^2 (1 + \sqrt{6}\omega) \right) \Phi_{,w} \\ & - \left( \beta^2 + \sqrt{6}\beta\omega^2 - 6\gamma^2\omega^2 e^{-2\omega w} - 12\omega^2 V_0 e^{-6\omega w} \right) \Phi. \end{aligned} \quad (45)$$

If we take  $\lambda = \pm\sqrt{6}$ , i.e.  $\omega^2 = \frac{1}{6}$ , this equation admits an exact solution as

$$\Phi^+(w) = \Phi_0 \exp \left( \frac{(\sqrt{6}\beta + 6\beta^2)w + 3\sqrt{6}\gamma^2 e^{-\frac{\sqrt{6}}{3}w} + 2\sqrt{6}V_0 e^{-\sqrt{6}w}}{2(\sqrt{6} + 6\beta)} \right), \quad \lambda = \sqrt{6}, \quad (46)$$

$$\Phi^-(w) = \Phi_0 \exp \left( \frac{(\sqrt{6}\beta + 6\beta^2)w - 3\sqrt{6}\gamma^2 e^{\frac{\sqrt{6}}{3}w} - 2\sqrt{6}V_0 e^{\sqrt{6}w}}{12\beta} \right), \quad \lambda = -\sqrt{6}. \quad (47)$$

We note that when  $\gamma = 0$ , the solution (44) is the invariant solution of the Lie algebra  $A_{(iii)}$ .

### 3.2.2. Case B: Lie invariants of $G_2$

In this section we consider again the exponential potential (33) but now we assume the relation  $\lambda = \frac{1}{\omega}$  between the constants  $\lambda$  and  $\omega$ . In this case the WDW equation (32) is invariant under the action of the Lie algebra  $G_2$ . From table I, we see that there exist the following two dimensional solvable subalgebras:  $A_{(i)}$ ,  $A_{(ii)}$ ,  $B_{(i)} = \{X_2, X_3\}$  and  $B_{(ii)} = \{X_2 + \beta X_\Psi, X_3\}$  for arbitrary values of  $\lambda$  and when  $\lambda^2 = 6$  the extra subalgebras  $A_{(iii)}$ ,  $A_{(vi)}$ ,  $B_{(iii)} = \{X_2, X_3 + \delta X_\Psi\}$  and  $B_{(iv)} = \{X_2 + \beta X_\Psi, X_3 + \delta X_\Psi\}$  will be added to the above list.

For arbitrary constant  $\lambda$ , the application of  $A_{(i)}$  and  $A_{(ii)}$  is studied before, so we will investigate the reduction with the more general subalgebra  $B_{(ii)}$  application of which on (32) results

$$\Psi_{B_{(ii)}}(a, \phi, \zeta) = a^{6\beta\omega - 2\rho} (6a^2 + (6\omega^2 - 1) e^{2\omega\phi} \zeta^2) \Phi(w), \quad w = \phi - 6\omega \ln a, \quad (48)$$

where  $\rho = \frac{12\beta\omega + \sqrt{6}\omega + 1}{2(1 - 6\omega^2)}$ ,  $\omega^2 \neq \frac{1}{6}$  and  $\Phi(w)$  satisfies the second order ODE

$$\begin{aligned} 0 = & (1 - 6\omega^2) \Phi_{,ww} + 6\omega (12\beta\omega + \sqrt{6}\omega + 1) \Phi_{,w} + \\ & + (12V_0 - (1 + 6\beta\omega) (1 + 6\beta\omega + \sqrt{6}\omega) e^{\frac{w}{\omega}}) \Phi. \end{aligned} \quad (49)$$

In what follows some solutions are presented for the special value  $\lambda^2 = 6$ .

*a. Subcase B1:  $\lambda^2 = 6$*  As we have mentioned above when  $\lambda = \pm\sqrt{6}$ , i.e.  $\omega = \pm\frac{\sqrt{6}}{6}$ , we may study the reduction of equation (32) by using the Lie subalgebra  $B_{(iv)}$ . Thus, we apply  $B_{(iv)}$  on (32) to obtain the wave functions as:

$$\Psi_{+B_{(iv)}} = a^{\sqrt{6}\beta} \exp \left( - \frac{(\sqrt{6}\beta\zeta + \zeta - 6\delta) z e^{\frac{\sqrt{6}}{3}\phi}}{6a^2} \right) \Phi_+(w_-), \quad w_- = \phi - \sqrt{6} \ln a, \quad (50)$$

with

$$\Phi_+(w_-) = \Phi_{0+} \exp \left[ \frac{\sqrt{6}}{2(\sqrt{6} + 6\beta)} \left( 2V_0 e^{-\sqrt{6}w_-} - 3\delta^2 e^{\frac{\sqrt{6}}{3}w_-} \right) + \frac{3\beta + \sqrt{6}}{6} w_- \right], \quad (51)$$

for  $\lambda = \sqrt{6}$  and

$$\Psi_{-B_{(iv)}} = a^{-\sqrt{6}\beta} \exp \left( \frac{(\sqrt{6}\beta\zeta + 6\delta) e^{-\frac{\sqrt{6}}{3}\phi}}{6a^2} \right) \Phi_-(w_+), \quad w_+ = \phi + \sqrt{6} \ln a, \quad (52)$$



with

$$\Phi_{-}(w_{+}) = \Phi_{0-} \exp \left[ -\frac{\sqrt{6}}{12\beta} \left( 2V_0 e^{\sqrt{6}w_{+}} - 3\delta^2 e^{\frac{\sqrt{6}}{3}w_{+}} \right) + \frac{(6\beta - \sqrt{6})}{12} w_{+} \right], \quad (53)$$

and for  $\lambda = -\sqrt{6}$ . Finally we would like to emphasize that since the WDW equation is linear, the general invariant solution with the potential (33) with  $\lambda^2 = 6$  and  $\omega = \frac{1}{\lambda}$  is

$$\Psi_{\Sigma}(a, \phi, \zeta) = \sum_{\bar{\beta}, \bar{\gamma}} \bar{\Psi}_{\pm A_{(iv)}} + \sum_{\beta, \delta} \Psi_{\pm B_{(iv)}}, \quad (54)$$

where  $\bar{\Psi}_{\pm A_{(iv)}}$  are the invariant solution (44) with respect to the subalgebra  $A_{(iv)}$ .

### 3.2.3. Case C: Lie invariants of $G_3$

Consider now the special case where  $\lambda = 4\omega$ , in the exponential potential (33), and  $\omega = \frac{\sqrt{6}}{3}$ . In that case the WDW equation (32) is invariant under the group of transformation which form the Lie algebra  $G_3$ . We apply the extra subalgebras  $C_{(i)} = \{X_1 + \gamma X_{\Psi}, X_4 + \delta X_{\Psi}\}$ ,  $C_{(ii)} = \{X_2 + \beta X_{\Psi}, X_3\}$ ,  $C_{(iii)} = \{X_3, X_4 + \rho X_{\Psi}\}$ ,  $C_{(iv)} = \{X_2 + \beta X_{\Psi}, X_3 + cX_4\}$

From the Lie algebra  $C_{(i)}$  we find that,

$$\Psi_{C_{(i)}}(a, \phi, \zeta) = e^{-\frac{\sqrt{6}}{6}\phi} e^{\gamma\zeta} \exp \left( \frac{\sqrt{6}\gamma^2}{6\delta} a^3 e^{-\frac{\sqrt{6}}{2}\phi} - \frac{\sqrt{6}\delta}{2a} e^{-\frac{\sqrt{6}}{6}\phi} + \frac{a^7}{14\delta} e^{-\frac{7\sqrt{6}}{6}\phi} \right), \quad (55)$$

while the Lie algebra  $C_{(ii)}$  give us the solution

$$\Psi_{C_{(ii)}} = \Phi(u) e^{-3\beta\phi} a^{3\sqrt{6}\beta}, \quad (56)$$

where  $\Phi(u)$  is given by the following equation

$$u\Phi_{,uu} + \left( 10 + \frac{3\sqrt{6}}{7}\beta \right) \Phi_{,u} + \frac{3}{28}\Phi = 0 \quad (57)$$

and  $u = a^4 \left( 2a^2 e^{\frac{2}{3}\sqrt{6}\phi} + \zeta^2 e^{-\frac{2}{3}\sqrt{6}\phi} \right)$ .

Furthermore, from the application of  $C_{(iii)}$ , we find

$$\Psi_{C_{(iii)}} = \exp \left( \frac{\sqrt{6}}{84\beta} \left( 6e^{-\frac{7\sqrt{6}}{6}v} + 7\beta v \right) \right) \exp \left( -\frac{\sqrt{6}\beta}{4a^3} z \right) \quad (58)$$

where  $z = \left( 2a^2 e^{-\frac{\sqrt{6}}{6}\phi} + \zeta e^{\frac{\sqrt{6}}{2}\phi} \right)$ , and  $v = \phi - \sqrt{6} \ln a$ . From the rest of the Lie subalgebras of  $G_5$ , we derive trivial solutions for the wavefunction  $\Psi$ .

As we discussed above, it is possible to determine invariant solution by applying the zero-order invariants of only one of the Lie symmetry vectors. For example, the application of the Lie symmetry vector  $X_4$ , in (32), reduce the WDW equation to the following second order differential equation

$$\Phi_{,\zeta\zeta} + 6\Phi e^{-\frac{2\sqrt{3}}{3}v} = 0, \quad (59)$$

where  $\Psi(a, \phi, \zeta) = \Phi(v, \zeta)$ , and  $v = \phi - \sqrt{6} \ln a$ . From the last we find the solution

$$\Phi(v, \zeta) = \Phi_1(v) \sin \left( e^{-\frac{2\sqrt{3}}{3}v} \right) + \Phi_2(v) \sin \left( e^{-\frac{2\sqrt{3}}{3}v} \right). \quad (60)$$

Below we continue with the determination of the analytical solution of the field equations for that model.

#### 4. CONSERVATION LAWS AND ANALYTICAL SOLUTIONS OF THE FIELD EQUATIONS

In [16], it has been shown how to extract the conservation laws and compute the Noether integrals for a given classical Lagrangian from the Lie symmetries of the WDW equation. In this section, for each Lie group of transformations in terms of which the WDW equation (32) is invariant, we are going to find the conservation laws correspond to the classical Lagrangian (12) and the corresponding lapse functions  $N(a, \phi, \zeta)$ .

To start let us first consider the Lie algebra  $G_1$ . In this case Lagrangian (12) has the following integrals

$$I_1 = \frac{1}{N} a e^{2\omega\phi} \dot{\zeta}, \quad \text{with } N = N_1(a, \phi, \zeta), \quad (61)$$

$$I_2 = \frac{1}{N} \left( \lambda a^2 \dot{a} + a^3 \dot{\phi} + a \zeta e^{2\omega\phi} \left( \frac{\lambda}{6} - \omega \right) \dot{\zeta} \right), \quad \text{with } N = N_2(a, \phi, \zeta), \quad (62)$$

where  $N_1(a, \phi, \zeta) = N_1(a, \phi) e^{-2\mu\zeta}$  and  $N_2(a, \phi, \zeta) = N_2\left(\phi - \frac{6}{\lambda} \ln a, \zeta a^{-\frac{\lambda-6\omega}{\lambda}}\right) a^{3-\frac{12\mu}{\lambda}}$ , and  $\mu \in \mathbb{R}$ . However, when  $\lambda = \frac{1}{\omega}$ , from the Lie algebra  $G_2$  we have an extra conservation law which comes from the Lie symmetry  $X_3$ , that is

$$I_3 = \frac{1}{N} \left( -2a^2 \zeta \dot{a} + 2\omega a^3 \zeta \dot{\phi} + \left( \frac{(1-6\omega^2)}{6} a e^{2\omega\phi} + a^3 \right) \dot{\zeta} \right), \quad \text{with } N = N_3(a, \phi, \zeta), \quad (63)$$

where  $N_3(a, \phi, \zeta) = N\left(\phi - 6\omega \ln a, \frac{e^{-2\omega\phi}}{6\omega^2-1} a^{6\omega^2-1} [(6\omega^2-1)\zeta^2 e^{2\omega\phi} + 6a^2]\right)$ .

It should be noted that the integrals  $\{I_1, I_2, I_3\}$  satisfy a Poisson algebra which has the same structure as the commutative algebra of the symmetry vectors  $\{X_1, X_2, X_3\}$ . This means that the dynamical system which is described by Lagrangian (12) admits three independent conservation laws. On the other hand, since (12) is "time-independent", the Hamiltonian constraint (15), i.e.  $H = h = 0$ , shows itself also as a conservation law which here is nothing but the first Friedmann equation (15). It is easy to see that when  $\lambda = 6\omega$ , the dynamical system with Lagrangian (12) is Liouville integrable with respect to the integrals  $\{h, I_1, I_2\}$ . Moreover, when  $\omega^2 = \frac{1}{6}$ , this Lagrangian admits the Noether integrals  $\{h, I_1, I_2, I_3\}$ . Finally when  $\lambda = 4\omega$ , and  $\omega = \frac{\sqrt{6}}{6}$ , then from the elements of the Lie algebra  $G_3$ , new Noetherian conservation laws arise, it is easy to see that in that case the field equations form a Liouville integrable system.

Below we determine the analytical solution of the field equations for the case where  $\lambda = 6\omega$ .

##### 4.1. Analytical solution of the field equations for $\lambda = 6\omega$

In order to determine the exact solution of the field equations we will use the Hamilton-Jacobi method to reduce the Hamiltonian system. For the Hamiltonian (15) with the potential function (33) if we choose  $\lambda = 6\omega$ , the (null) Hamilton-Jacobi equation takes the form

$$-\frac{1}{12a} \left( \frac{\partial S}{\partial a} \right)^2 + \frac{1}{2a^3} \left( \frac{\partial S}{\partial \phi} \right)^2 + \frac{1}{2a e^{2\omega\phi}} \left( \frac{\partial S}{\partial \zeta} \right)^2 + V_0 a^3 e^{-6\omega\phi} = 0, \quad (64)$$

where  $S = (a, \phi, \zeta)$  and  $p_a = \frac{\partial S}{\partial a}$ ,  $p_\phi = \frac{\partial S}{\partial \phi}$ ,  $p_\zeta = \frac{\partial S}{\partial \zeta}$ . To go forward, let us introduce the following coordinate transformations

$$a = e^{\omega x}, \quad \phi = x + y, \quad \zeta = \zeta. \quad (65)$$

In terms of these new variables the Hamiltonian (15) becomes

$$0 = -\frac{e^{-3\omega x}}{6\omega^2} [p_x^2 - 2p_x p_y + (1-6\omega^2) p_y^2 - e^{-2y} p_\zeta^2] + 2V_0 e^{\omega(3x-6y)}. \quad (66)$$

Also, the Noether conservation laws (61) and (62) are

$$I_1 = p_\zeta, \quad I_2 = p_x. \quad (67)$$

Now, one is led to the following field equations

$$\dot{x} = -\frac{1}{6} (Ne^{3\omega x})^{-1} \omega^2 (p_x - p_y), \quad (68)$$

$$\dot{y} = \frac{1}{6} (Ne^{3\omega x})^{-1} (p_x + (6\omega^2 - 1) p_y), \quad (69)$$

$$\dot{\zeta} = p_\zeta (Ne^{3\omega x + 2\omega y})^{-1}. \quad (70)$$

Furthermore, the Hamilton-Jacobi equation (64) in the new coordinates (65) takes the form

$$0 = -\frac{e^{-3\omega x}}{6\omega^2} \left[ \left( \frac{\partial S}{\partial x} \right)^2 - 2 \left( \frac{\partial S}{\partial x} \right) \left( \frac{\partial S}{\partial y} \right) + (1 - 6\omega^2) \left( \frac{\partial S}{\partial y} \right)^2 - e^{-2\omega y} \left( \frac{\partial S}{\partial z} \right)^2 \right] + 2V_0 e^{\omega(3x-6y)}.$$

By using the integrals (67), we may separate the action function  $S(x, y, \zeta)$  as  $S(x, y, \zeta) = S_1(x) + S_2(y) + S_3(\zeta)$ , where

$$S_1(x) = I_2 x + x_0, \quad S_3(\zeta) = I_1 \zeta + \zeta_0, \quad (71)$$

and

$$S_2(y) = -\frac{I_2 y}{6\omega^2 - 1} - \left( \frac{\sqrt{6}}{6\omega^2 - 1} \right) \int \sqrt{\omega^2 I_2^2 - I_1 \omega^2 (6\omega^2 - 1) e^{-2\omega y} - 2V_0 \omega^2 (6\omega^2 - 1) e^{-6\omega y}} dy, \quad (72)$$

for  $\omega^2 \neq \frac{1}{6}$  and

$$S_2(y) = \frac{I_2}{2} y \pm \frac{V_0}{6I_2} e^{-\sqrt{6}y} \pm \frac{\sqrt{6}I_1^2}{4I_2} e^{-\frac{\sqrt{6}}{3}y}, \quad (73)$$

for  $\omega^2 = \frac{1}{6}$ . Therefore, the field equations will be reduced to the following system

$$\dot{x} = -\frac{1}{6} (Ne^{3\omega x})^{-1} \omega^2 \left( I_2 - \left( \frac{\partial S}{\partial y} \right) \right), \quad (74)$$

$$\dot{y} = \frac{1}{6} (Ne^{3\omega x})^{-1} \left( I_2 + (6\omega^2 - 1) \left( \frac{\partial S}{\partial y} \right) \right), \quad (75)$$

$$\dot{\zeta} = I_1 (Ne^{3\omega x})^{-1} e^{-2\omega y}. \quad (76)$$

To linearize the system (74)-(76), we consider the conformal transformation  $dt = (Ne^{3\omega x}) d\tau$ . In terms of the evolution parameter  $\tau$  the above system will be

$$\frac{dx}{d\tau} = \frac{\omega^2}{6} (p_y - I_2), \quad \frac{dy}{d\tau} = \frac{1}{6} (I_2 + (6\omega^2 - 1) p_y), \quad \frac{d\zeta}{d\tau} = I_1 e^{-2\omega y}. \quad (77)$$

Here we note that when  $I_1 = 0$ , then  $\zeta = \text{const}$ , in that case the action integral (10) reduced to that of the scalar field cosmology.

In figure 1, we give the qualitative behavior of the scale factor  $a(t)$ , which follows from the numerical solution of the system, (74)-(76), for the proper time,  $N = 1$ , and for  $\omega = 10^{-1}$ . The plots are for different values of the constant  $I_1$ . We observe that the existence of the vector field  $\zeta$ , i.e.  $I_1 \neq 0$ , and for large values of the constant  $I_1$ , in the early universe the scale factor run faster. In figure 2 we give the qualitative behavior of the scalar field  $\phi(a)$ , where we show that the behavior of the scalar field is independent on the existence of the vector field. However what it changes is the minimum value of the field,  $\min \phi$  in which increase as the parameter  $I_1$  increase.

In the simplest case where  $\omega = \frac{\sqrt{6}}{6}$  we obtain the closed-form solution

$$x(\tau) = \frac{\sqrt{6}V_0}{36I_2^2} x_0 e^{-\frac{\sqrt{6}}{6}I_2\tau} + \frac{\sqrt{6}}{24I_2^2} \left( x_0 e^{-\frac{\sqrt{6}}{6}I_2\tau} \right)^{\frac{1}{3}} - \frac{I_2}{72}\tau, \quad (78)$$

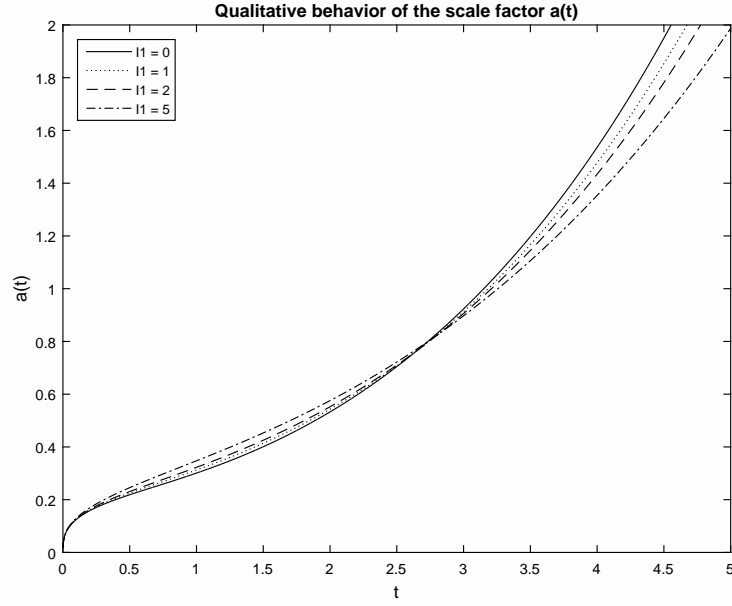


FIG. 1: Qualitative behavior of the scale factor  $a(t)$ , given by the solution of the system (74)-(76), for the proper time,  $N = 1$ . For the plot we considered the initial condition  $a(0^+) \rightarrow 0$ , whereas for the parameters we select,  $\omega = 10^{-1}$ ,  $I_2 = -2$ ,  $V_0 = 0.5$ . The solid line is for  $I_1 = 0$ , which corresponds to scalar field cosmology without the vector field, the dot line is for  $I_1 = 1$ , the dashed line is for  $I_1 = 2$ , and the dash-dot line is for  $I_1 = 5$ .

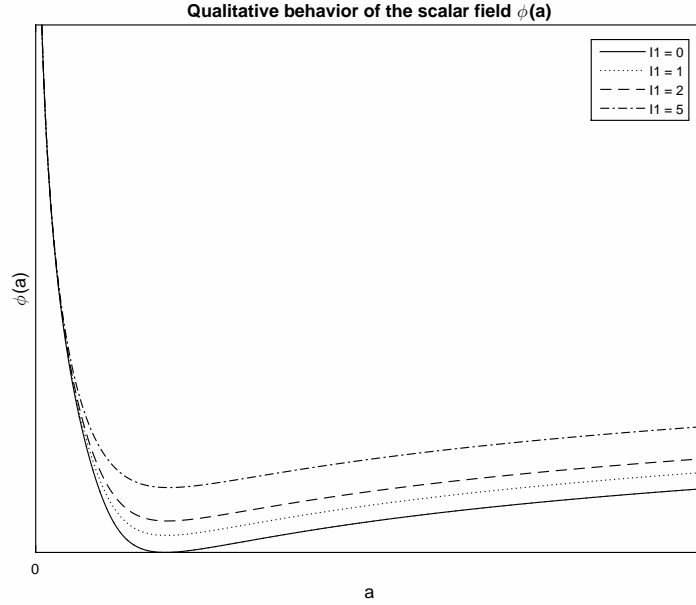


FIG. 2: Qualitative behavior of the scalar field  $\phi(a)$ , given by the solution of the system (74)-(76), for the proper time,  $N = 1$ . The values of the free parameters are that of figure 1.

$$y(\tau) = \frac{I_2}{6}\tau - \frac{\sqrt{6}}{6} \ln x_0, \quad \zeta(\tau) = -\frac{3\sqrt{6}I_1}{I_0} \left( x_0 e^{-\frac{\sqrt{6}}{6}I_2\tau} \right)^{\frac{1}{3}} + \zeta_0. \quad (79)$$

Finally, by recalling the scale factor  $a(\tau) = \exp(\omega x(\tau))$ , we find the following form for the line element of the FRW space-time

$$ds^2 = -\exp(6\omega x(\tau)) d\tau^2 + \exp(2\omega x(\tau)) (\delta_{ij} dx^i dx^j). \quad (80)$$

where  $x(\tau)$  is given by (78).

## 5. SUMMARY

In this paper, we studied a scalar-vector field model of cosmology in a Lie symmetry point of view. In the presented action, in addition to a minimally coupling between the scalar field and gravity, there is also a coupling between the scalar and the kinetic energy term of the vector field. For the background geometry, we have considered a flat FRW metric and then set up the phase space by taking the scale factor  $a$ , scalar field  $\phi$  and the vector field  $\zeta$  as the independent dynamical variables. The Lagrangian of the model in the configuration space spanned by  $\{a, \phi, \zeta\}$  is so constructed that its variation with respect to these dynamical variables yields the Einstein field equations. Therefore, the Lagrangian of the cosmological model forms a three dimensional Hamiltonian system with two unknown functions, one of which,  $f(\phi)$ , denotes the interaction between the scalar and the vector fields while another,  $V(\phi)$ , is the potential function of the scalar field. To determine these unknown functions we have used a geometric criterion known as the Lie symmetry method according to which a PDE (in our case the WDW equation) will be invariant under the act of the generators of a Lie algebra. Since the Lie symmetries of the WDW equation are related with the conformal algebra of the corresponding minisuperspace, we showed that the function  $f(\phi)$  is of the form of an exponential function if the minisuperspace is conformally flat, this also is the only case in which the WdW equation can admit Lie point symmetries. Also, our symmetry considerations result an exponential potential function.

This approach is also a powerful tool in finding the classical solutions to a given Lagrangian, including the one presented above. In this approach, one is concerned with finding the cyclic variables related to conserved quantities and consequently reducing the classical dynamics of the system to a manageable one. Indeed, the existence of Lie symmetry means that phase flux is conserved along a vector field  $X$  and thus a constant of motion exists.

In this set-up we studied the closed-form solutions of the WDW and the classical field equations for a scalar-vector field cosmological model. To do this, we applied the Lie invariants to find the invariant solutions of the WDW equation. As we mentioned above, the importance of the application of Lie symmetries is that they can be used in order to construct conservation laws (Noether symmetries) for the classical field equation. Hence, we were able to study the integrability of the field equations and to find classical closed-form solutions. This method is related with the application of Noether's theorem in cosmological models; however, as we have showed, it is a more general selection rule, since, we are able to relate the classical exact solution with the invariant form of the wave function of the universe.

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